A Micro-Course in Symbolic Logic V-4

Definition: A **STATEMENT** is an oral or written assertion that must be either true or false, but not both.
Definition: An **ARGUMENT** is an assertion that a statement (called the conclusion) follows from (i.e. is implied by) another set of statements (called the premises).

Logical connectives are defined by their truth tables as below:

AND (Conjunction)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

OR (Disjunction, Inclusive "or")

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

NOT (Negation)

p	~ p
T	F
F	T

IF…THEN (Conditional)

p	q	$p \rightarrow q$	
T	T	T	
T	F	F	
F	T	T	} Paradox of
F	F	T	} the Conditional

IF AND ONLY IF - "IFF" (Biconditional)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Definitions of additional terms:

- A CONTRADICTION is a statement (form) that is always false. (Its truth table has all F's).
- A TAUTOLOGY is a statement (form) that is always true. (Its truth table has all T's).
- IMPLICATION: A statement P IMPLIES a statement Q iff $P \rightarrow Q$ is a tautology. We write $P \Rightarrow Q$.
- EQUIVALENCE: A statement P is EQUIVALENT to a statement Q iff $P \leftrightarrow Q$ is a tautology. We write $P \Leftrightarrow Q$ or $P \equiv Q$.
- VALIDITY: An argument is valid iff the conjunction of the premises implies the conclusion.

There are FOUR FORMS of PROOF that an argument is valid. Here are logical steps for the three forms we will use most.

Direct Proof: (a) Assume the premises to be true. (b) By logical means, arrive at the conclusion.
Contrapositive Proof: (a) Assume the negation of the conclusion. (b) By logical means, arrive at the negation of the conjunction on the premises.
Proof by Contradiction: (a) Assume the premises and the negation of the conclusion to be true. (b) By logical means, arrive at any contradiction.
There is also a fourth form. **Proof by Induction:** There are three forms of induction. A truth table proof is an example of "perfect induction" in which the argument is shown to be valid by verification for all possible values of the premises. The other two, induction on the integers and transfinite induction, are not considered in this lesson.

To initiate development of our repertoire of "logical means", we can use truth tables to show the validity of seven basic arguments and three equivalence laws.

DeMorgan's Laws: (a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$ (b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Conditional Equivalence: $p \to q \equiv \sim p \vee q$

[For simplicity, we will assume ,without need for justification, the commutative properties of $\wedge$ and $\vee$. For example, $\sim p \vee q \equiv q \vee \sim p$.]

Our seven basic valid arguments are called Modus Ponens, Modus Tolens, Hypothetical Syllogism, Disjunctive Syllogism, Addition, Simplification and Conjunction:

M.P. $\begin{array}{l} P \to Q \\ \underline{P \qquad} \\ \therefore Q \end{array}$

M.T. $\begin{array}{l} P \to Q \\ \underline{\sim Q \qquad} \\ \therefore \sim P \end{array}$

H.S. $\begin{array}{l} P \to Q \\ \underline{Q \to R \quad} \\ \therefore P \to R \end{array}$

D.S. $\begin{array}{l} P \vee Q \\ \underline{\sim P \quad} \\ \therefore Q \end{array}$

ADD: $\begin{array}{l} \underline{\quad P \qquad} \\ \therefore P \vee Q \end{array}$

SIMP: $\begin{array}{l} \underline{P \wedge Q} \\ \therefore P \end{array}$

CONJ: $\begin{array}{l} P \\ \underline{Q \qquad} \\ \therefore P \wedge Q \end{array}$

As a first exercise, prove each of these laws and arguments with truth tables (perfect induction). For example, draw the truth table for $[(P \to Q) \wedge P] \to Q$. If you get all T's, you have shown this statement to be a tautology. That, in turn, means the conjunction of the premises of Modus Ponens implies the conclusion. So you have shown that M.P. is a valid argument.

Exercise Your Logic

Part I Using the seven basic valid arguments and three equivalences, provide a Direct Proof of each of these:

1.
$A \to D$
$\sim(D \vee C)$
$\underline{A \vee B /\ \therefore B}$

2.
$(A \vee B) \to C$
$E \to F$
$\sim A \to E$
$\underline{K \wedge \sim F /\ \therefore C}$

3.
$(A \vee B) \to \sim C$
$A \vee K$
$\underline{\sim K /\ \therefore \sim(C \vee K)}$

4.
$F \to \sim G$
$\sim F \to (H \to \sim G)$
$(\sim I \vee \sim H) \to G$
$\underline{\sim I /\ \therefore \sim H}$

5.
$(A \to B) \to (C \vee D)$
$(C \vee D) \to K$
$K \to E$
$\underline{\sim E \wedge \sim A /\ \therefore \sim(A \to B)}$

Part II Using the seven basic valid arguments and three equivalences, provide a Proof by Contradiction of each of these:

1.
$M \to N$
$\sim(N \vee C)$
$\underline{M \vee B /\ \therefore B}$

2.
$Q \to R$
$\sim S \to (T \to U)$
$S \vee (Q \vee T)$
$\underline{\sim S /\ \therefore R \vee U}$

3.
$P \to Q$
$Q \to \sim(P \vee R)$
$\underline{P /\ \therefore K}$

4.
$(\sim P \vee Q) \wedge (P \vee R)$
$\underline{\sim(Q \vee R) /\ \therefore \sim(Q \wedge R)} \wedge \sim K$

Part III Prove that this argument is invalid.

$(P \vee Q) \to R$
$\underline{R /\ \therefore P \vee Q}$